Module-1.Ordinary Differential Equation

Dr. Chandra Shekher Singh

Department of Basic Science UPTTI,
Kanpur-208001, India

Branch-TT,TE , Apr. 05, 2020-SEM-II

Outline of the Presentation

- Variation of Parameters
- Working rule for Solving $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ by Variation of Parameters, where P, Q and R are functions of x or Constants.
- Based Examples
- Based Questions

Variation of Parameters

Definition

The Wornskian of n functions $y_1(x), y_2(x), ..., y_n(x)$ is denoted by $W(y_1, y_2, ..., y_n)$ and is defined to be the dterminant

$$W(y_1, y_2, ..., y_n) = W(x) = \begin{vmatrix} 1 & y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ 1 & y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & y_1'(x) & y_2'(x) & \dots & y_n'(x) \end{vmatrix}.$$

Examples

1. (Example-1) Consider the two function $f_1(x) = x^3$ and $f_2(x) = x^2$, find Wronskian i.e, $W(f_1, f_2)$ (The solution is given below).
2. (Example-2) Consider the two function $f_1(x) = \sin(x)$ and $f_2(x) = \cos(x)$, find Wronskian i.e, $W(f_1, f_2)$ (The

Variation of Parameters

Definition

Variation of Parameters: Variation of Parameters is a method for producing a particular solution to an nonhomogeneous equation by exploiting the (Usually much simpler to find) solutions to the associated homogeneous equation.

Working Procedure for solving

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

by Variation of Parameters, where P, Q and R are functions of x or Constants.

Step-1: Re-write the given equation as

$$y_2 + Py_1 + Qy = R \quad (1)$$

Step-2:

$$y_2 + Py_1 + Qy = 0 \qquad (2)$$

which is obtained by taking $R = 0$ in (1). Solve (2) by method of previous chapter as the case may be. Let the general solution of (2) i.e., C.F. of (1) be

$$y = C_1u + C_2v. \qquad (3)$$

where, C_1, C_2 being arbitrary constants.
Step-3: General solution of (1) is

$$y = C.F. + P.I. \qquad (4)$$

where,

$$C.F. = C_1 u + C_2 v. \tag{5}$$

, C_1, C_2 being arbitrary constants.
and

$$P.I. = uf(x) + vg(x). \tag{6}$$

where,

$$f(x) = -\int \frac{vR}{W} dx \text{ and } g(x) = \int \frac{uR}{W} dx \tag{7}$$

where, dterminant

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}.$$

Apply the method of variation of parameters to solve

Examples

1. (Example-1) $x^2y_2 + 3xy_1 + y = \frac{1}{(1-x)^2}$ (The solution is given below).
2. (Example-2) $x^2y_2 + xy_1 - y = x^2 \log x$ (Do it yourself).

Example 1. Consider the two functions $f_1(x) = x^3$ and $f_2(x) = x^2$, find Wronskian i.e; $W(f_1, f_2)$.

Sol^n:- Given that

$f_1(x) = x^3$, and $f_2(x) = x^2$

We have

$$W(f_1, f_2) = \begin{vmatrix} f_1(x) & f_2(x) \\ \frac{d}{dx} f_1(x) & \frac{d}{dx} f_2(x) \end{vmatrix} = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix}$$

$$= f_1(x) \cdot \frac{d}{dx} f_2(x) - f_2(x) \frac{d}{dx} f_1(x)$$

$$= x^3 \cdot 2x - x^2 \cdot 3x^2$$

$$= 2x^4 - 3x^4 = -x^4 \ \#.$$

Example 2. Consider the two function $f_1(x) = \sin x$ and $f_2(x) = \cos x$, find Wronskian i.e, $W(f_1, f_2)$.

Sol^n:- Wronskian $W(f_1, f_2)$

$$= \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 x = -1 \ \#.$$

Apply the method of Variation of Parameters

Solve

(i) $x^2 y_2 + 3x y_1 + y = \frac{1}{(1-x)^2}$.

(ii) $x^2 y_2 + x y_1 - y = x^2 \log x$.

Solⁿ: (i) Step 1. the coefficient of highest order must be unity.

i.e, coefficient of y_2 must be unity.

so, $y_2 + \frac{3}{x} y_1 + \frac{1}{x^2} y = \frac{1}{x^2(1-x)^2}$

where $P = \frac{3}{x}$, $Q = \frac{1}{x^2}$, $R = \frac{1}{x^2(1-x)^2}$

Consider, $y_2 + \frac{3}{x} y_1 + \frac{1}{x^2} y = 0$

i.e, $x^2 y_2 + 3x y_1 + y = 0$

let $x = e^z \Rightarrow z = \log_e x$, $x^2 y_2 = D(D-1)$

$x y_1 = D$.

A.E. is $f(D) = 0$

i.e, $f(m) = 0$

Scanned with CamScanner

$\Rightarrow \quad m(m-1)+3m+1=0$

$\Rightarrow \quad (m+1)^2=0 \Rightarrow m=-1,-1$

$C.F. = (C_1 + zC_2)e^{-z}$

$= C_1 e^{-z} + C_2 z e^{-z}$

$= C_1 x^{-1} + C_2 x^{-1} \log_e x.$

, C_1 and C_2 are arbitray constants.

Let $u = x^{-1}$, $v = x^{-1}\log_e x$, $R = x^{-2}(1-x)^{-2}$

$$W(u,v) = \begin{vmatrix} x^{-1} & x^{-1}\log_e x \\ -x^{-2} & x^{-2} - x^{-2}\log x \end{vmatrix}$$

i.e,

$$W(u,v) = \begin{vmatrix} x^{-1} & x^{-1}\log_e x \\ \frac{d}{dx}x^{-1} & \frac{d}{dx}(x^{-1}\log x) \end{vmatrix} = x^{-3} \neq 0.$$

$$P.I. = u \cdot f(x) + v\, g(x)$$

where $$f(x) = -\int \frac{vR}{W}\, dx$$

$$= -\int \frac{x^{-1} \log x \cdot x^{-2}\, (1-x)^{-2}}{x^{-3}}\, dx$$

$$= -\int (1-x)^{-2} \log x\, dx$$

$$= -\left[\frac{1}{1-x} \log x - \int \frac{1}{x(1-x)}\, dx \right] \quad \text{I. by}$$

$$= -\frac{\log x}{1-x} + \int \left(\frac{1}{x} - \frac{1}{1-x} \right) dx$$

$$= \frac{-\log x}{1-x} + \log x - \log(1-x)$$

$$= -(1-x)^{-1} \log x + \log x - \log(1-x).$$

$$g(x) = \int \frac{uR}{W}\, dx$$

$$g(x) = \int \frac{x^{-1} x^{-2} (1-x)^{-2}}{x^{-3}} dx$$

$$= (1-x)^{-1}$$

$$P.I. = u \cdot f(x) + v \cdot g(x)$$

$$= x^{-1} \left\{ -(1-x)^{-1} \log x + \log x - \log(1-x) \right\}$$

$$+ x^{-1} \log x \, (1-x)^{-1}$$

$$= x^{-1} \log x - x^{-1} \log(1-x)$$

$$= x^{-1} \left[\log \frac{x}{1-x} \right]$$

Hence the general solution of given differenti equation is.

$$y(x) = C.F. + P.I.$$

$$= c_1 x^{-1} + c_2 x^{-1} \log x + x^{-1} \log \left\{ \frac{x}{1-x} \right\}.$$